PMM U.S.S.R., Vol.52, No.5, pp.603-611, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

DIFFRACTION OF A SUPERCOMPRESSED DETONATION WAVE REGULARLY REFLECTED FROM THE WALL OF AN OBTUSE WEDGE*

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The problem of the diffraction of a supercompressed detonation wave representing the regular reflection of a plane shock wave, by a corner of a wall turned by a small angle ε is solved. The properties of the medium, the intensity and the angle of incidence of the shock wave are assumed to be such that the front does not cause a detonation, while the reflected front initiates it and causes it to disappear. An exact solution is constructed differing from the solutions for the shock wave /1, 2/ only in more complicated expressions for the parameters appearing in it and expressed in terms of the parameters of the regular unperturbed reflection /3/.

The intensity of the incident front, which can be arbitrary when there is no heat supply, is related to the heat supply when it occurs, just as the other parameters, by the condition of admissibility of solutions of the given type. This is caused by the complications which arise whenever two small parameters are present simultaneously, the second of which may be represented by the degree of overcompression $\iota\,(0<\iota\leqslant 1)$. However, when ι is finite, the condition does not result in any constraints and the intensity remains arbitrary.

The solutions can be radically simplified for the values of the defining parameters which admit of solutions of the given type, by expanding them in series in powers of ι near the value $\iota=0$, even though they have no meaning for this value of ι. Only the linear terms are retained. The pressure along the front, its form and the pressure exerted on the wall, are given in a simple, easily perceived form, and the latter is represented by expressions of the same type as those for the diffraction of an acoustic wave (with additional terms).

The simplified solutions are also close to exact solutions for moderate and finite ι. They can therefore be regarded as approximate solutions for supercompressed detonation waves of any intensity.

The diffraction of a detonation wave in which the heat supply was a small quantity, was studied earlier in /4/.

1. The flow pattern.

A plane shock wave front propagating through a hot mixture of gases impinges at a finite angle α on a rigid wall and is reflected regularly from it. Combination in the reflected supercompressed detonation front takes place instantaneously. Such reflection was discussed in /3/. In order to be able to neglect the non-linear effects /5/ we shall assume that the magnitude of the angle α is outside the immediate neighbourhood of its maximum possible value for a regular reflection.

At the instant $t=0$ the point of reflection N (Fig.1) passes through the corner point H at which the wall changes its direction by a small angle $\varepsilon\,(\varepsilon>0$ if the corner is convex). The region of inhomogeneous flow $ABCDEF$ which lies behind the curved reflected front and is bounded by the front, the wall and the arcs of the Mach circle (whose centre lies at the particle appearing at the point N at $t=0$), begins to expand. The region of inhomogeneous flow lies next to the homogeneous flows and is separated from them by the arcs CD and AF (or AG and GF). The point H may appear within this region, as well as outside it (H'). In the latter case a region of homogeneous flow, namely of the supersonic flow past the corner in the wall FGH', lies next to it. The corresponding cases of diffraction are called the subsonic and supersonic cases.

The motion in question represents a small perturbation of the homogeneous stream of the reacting mixture of gases behind a plane reflected front inclined to the wall at a finite angle γ. The linearized boundary conditions are referred to the contour $ABCDEF$ containing the segment ABC of the unperturbed reflected front.

The regions of various homogeneous flows are indicated in Fig.1 by the numbers 0 - 4, and the pressure p, density ρ, velocity of sound a, and the polytropic exponent $\varkappa$ relating to those regions are marked by the corresponding subscripts ($\varkappa_0=\varkappa_1$, $\varkappa_4=\varkappa_3=\varkappa_2$). We denote the velocities of the incident and reflected front relative to the media in front of them, by

*Prikl.Matem.Mekhan., 52, 5, 773-782, 1988

$U_0 = M_0 a_0$ and $U_D = M_D a_1 = a_1 (\varkappa_2/\varkappa_1)^{1/2} M_d$.

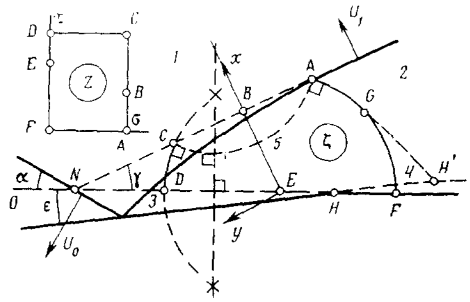

Fig.1

The evolution of heat in a detonation front is characterized by the quantity Q, in terms of which we express the heat generating capacity of the gaseous mixture $a_0^2 Q/(\varkappa_1 - 1)$ and velocity $U_J = a_1 M_J$ of propagation of the detonation front in the Chapman-Jouget mode, and

$$M_J = Q_0^{1/2} + \left(Q_0 + \frac{\varkappa_2}{\varkappa_1}\right)^{1/2} = \left(\frac{\varkappa_2}{\varkappa_1}\right)^{1/2} M_j \tag{1.1}$$

$$Q_0 = \frac{1}{2}\,\frac{\varkappa_2+1}{\varkappa_1-1}\left[(\varkappa_2-1)\,Q\,\frac{a_0^2}{a_1^2} - \frac{\varkappa_1-\varkappa_2}{\varkappa_1}\right]$$

Let us introduce the parameter ι representing the degree of supercompression of the detonation wave

$$\iota^4 = (M_d^2 - M_j^2)(M_d^2 - M_j^{-2})(M_d^2 - 1)^{-2} \tag{1.2}$$

When $\varkappa_1 = \varkappa_2$, (1.2) is reduced to $\iota^4 = 1 - Q/Q_J$ where Q_J corresponds to $\iota = 0$.

The defining parameters do not include a characteristic linear dimension, and the motion is selfsimilar.

If $\varkappa_1 = \varkappa_2$, we have $\iota = 1$ when $M_J = 1$, and the detonation wave beomes simply a shock wave, while when $M = M_J$, $\iota = 0$, it becomes the Chapman-Jouget wave.

The laws of conservation at the surfaces of discontinuity for different values of the polytropic exponent on different sides /6/, make it possible to write the relation connecting the values of the parameters of the flows in regions *1* and *2* (V_1, V_2 are the gas velocities behind the fronts relative to those in front of the fronts)

$$\frac{p_2}{p_1} = 1 + \frac{\varkappa_2}{\varkappa_2+1}(M_d^2 - 1)(1 + \iota^2), \tag{1.3}$$

$$\frac{\rho_1}{\rho_2} = 1 - \frac{1}{\varkappa_2+1}(1 - M_d^{-2})(1 + \iota^2)$$

$$V_2 = \frac{U_D}{\varkappa_2+1}(1 - M_d^{-2})(1 + \iota^2), \quad W_2 = V_2\,\frac{\cos(\beta+\gamma)}{\sin\beta}$$

In special cases the formulas yield relations connecting the parameters of the flows in regions O and *1*. The formulas are obtained at $\iota = 1$ by replacing the subscripts 2 by 1, 1 and D by O, the sum $\beta + \gamma$ by α; W_1, and W_2 are the stream velocities in the regions *1* and *2* relative to the point of reflection N; β is the angle by which the flow changes its direction during the passage through the incident front. The velocity U_D is expressed in terms of the angle γ: $U_D = W_1 \sin(\beta + \gamma)$ and the angle γ is found from the cubic equation /3/. The velocity of the flow in region *2* relative to the wall is equal to

$$V_W = a_2 M_W, \quad M_W = M_0\,(a_0/a_2)\operatorname{cosec}\alpha - M, \quad M = W_2/a_2 \tag{1.4}$$

The above expression uses the positions of the points *H* and *H'* to determine the subsonic and supersonic case of diffraction, and in the latter case also the acute angle θ_G' between the radius *EG* and the wall $\sec\theta_G' = M_W$, $M_W > 1$. If $\iota = 1$ and $\varkappa_1 = \varkappa_2 = 1{,}4$, then, as was shown in /2/, the point *G* cannot reach the point *A*. When $\iota < 1$ and $\varkappa_1 = \varkappa_2 < 1{,}4$, then as computations show, it is even more unlikely.

Fig.2 shows the boundaries of the supersonic cases. In the coordinates p_0/p_1, α with parameter $Q = 0.1; 1; 5; 20$, these are the regions APQ, BPQ, CPQ and DPQ. Every one of these regions is bounded by a line which branches off the corresponding boundary of regularity, and they all have a common segment of the boundary, i.e. the vertical segment *PQ* where $p_0/p_1 = 0$. We note that they correspond to the very strongly supercompressed waves only.

2. **The boundary value problem.** We attach the origin of the selfsimilar x, y coordinates to the point *E*, direct the x axis perpendicular to and y axis parallel to the

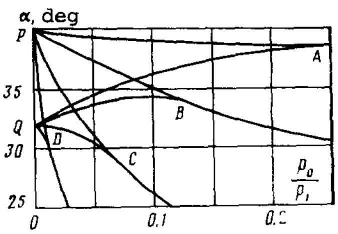

Fig.2

unperturbed reflected front. The coordinates are formed from the coordinates $X, Y: x = (X - V_2 t)/(a_2 t)$, $y = Y/(a_2 t)$ attached to the flow in front of the reflected front whose coordinate is

$$x = m = M \sin\gamma = \left(\frac{1 + \varkappa_2 M_d^2 - (M_d^2 - 1)\iota^2}{1 + \varkappa_2 M_d^2 + \varkappa_2 (M_d^2 - 1)\iota^2}\right)^{1/2} \tag{2.1}$$

The parameters of the inhomogeneous gas flow in region 5, which are indicated by a bar, are

$$\bar{p} = p_2 + p', \quad \bar{\rho} = \rho_2 + \rho', \quad \bar{u} = V_2 + u', \quad \bar{v} = v'$$

The primes denote perturbations, which can be replaced by the dimensionless functions

$$p = p'/(\rho_2 a_2^2), \quad \rho = \rho'/\rho_2, \quad u = u'/a_2, \quad v = v'/a_2$$

The equation describing the curved segment of the reflected front, taking (2.1) into account, will be written in the form $x = m + f(y)$ where f is a dimensionless function of order ε.

Below we shall assume for the time being, until the formulation has been completed and the solution of the problem constructed, that $\iota \neq 0$ is a finite quantity. Then, following the method used in /2/, we can obtain the following condition on the diffracting reflected front:

$$u = \frac{1}{\varkappa_2 + 1} \frac{M_d^2 + 1}{M_d^2} \frac{1 + \iota^2}{\iota^2} (f - yf') \tag{2.2}$$

$$v = -\frac{1}{\varkappa_2 + 1} \frac{a_1}{a_2} \frac{M_d^2 - 1}{M_d^2} M_D (1 + \iota^2) f'$$

$$p = \frac{1}{\varkappa_2 + 1} \frac{a_1}{a_2} \frac{\rho_1}{\rho_2} \left[2 + \frac{M_d^2 + 1 + (M_d^2 - 1)\iota^4}{M_d^2 \iota^2}\right] M_D (f - yf')$$

Using the same transformations as in /2/, we obtain from these conditions the relations

$$u = Ap, \quad y\partial v/\partial y = B\partial p/\partial y \tag{2.3}$$

$$A = \frac{1 - C}{m}, \quad B = \frac{\rho_2}{\rho_1} C, \quad C = \frac{(M_d^2 - 1)\iota^2}{M_d^2 + 1 + (M_d^2 - 1)\iota^2}$$

The system of equations of the unsteady, selfsimilar plane flow of gas is linearized with respect to the parameter ε and transforms, after eliminating the functions ρ, u, v, changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and carrying out the Buseman transformation $r = 2R/(1 + R^2)$, to the Laplace equation for the perturbation of the pressure p inside the unit circle. The curved segment AC of the detonation front maps onto an arc of the circle $2R\cos\theta = m(1 + R^2)$ shown in Fig.1 by the dashed line, orthogonal to the circle $R = 1$. The remaining elements of the boundary of the region of inhomogeneity do not change their configuration, but the correspondence of the points along the wall changes.

The resulting curvilinear rectangle with sides intersecting orthogonally, is mapped using the conformal transformation

$$z = \ln\frac{\zeta - \exp i\theta_2}{\zeta - \exp i\theta_1} - i\frac{\theta_2 - \theta_1}{2}$$

$$\theta_1 = \arcsin M^{-1} - \gamma, \quad \theta_2 = \pi - \arcsin M^{-1} - \gamma$$

into a rectangle in the plane $z = \sigma + i\tau$ (Fig.1)

$$0 < \sigma < l, \quad 0 < \tau < \pi \tag{2.4}$$

$$l = -\frac{1}{2}\ln q, \quad q = \frac{1 - \operatorname{tg}\gamma (M^2 - 1)^{1/2}}{1 + \operatorname{tg}\gamma (M^2 - 1)^{1/2}}$$

Here the bilinear function mapping the points marked with crosses in Fig.1 onto 0 and ∞, first maps the region into a semicircle, and then the logarithmic function maps it into a rectangle.

The points G and H in the z plane acquire the coordinates

$$\sigma_G = \frac{1}{2}\ln\frac{1 - \cos(\theta_G - \theta_2)}{1 - \cos(\theta_G - \theta_1)}, \quad \tau_G = 0$$

$$\sigma_H = 0, \quad \tau_H = \arccos\frac{1 + MM_W}{M + M_W}$$

The boundary conditions on the wall $(\partial p/\partial n = 0)$ and on the Mach arcs $(\partial p/\partial s = 0$ or a given pressure jump, and s is the coordinate along the contour) are the same as those in /2/.

We can describe the whole system of boundary conditions, just as in the case of a shock wave, by the single relation

$$P\,(\partial p/\partial\sigma) - Q\,(\partial p/\partial\tau) = S \tag{2.5}$$

in which, when $\sigma = l,\ 0 < \tau < \pi\ P/Q = b(\tau),\ S = 0;\ P = 1,\ Q = 0,\ S = \varepsilon M_W^2 (1 - M_W^2)^{-1/2}\delta(\tau - \tau_H) = s_-\delta(\tau - \tau_H)$ in the subsonic case $(M_W < 1)$, and $S = \varepsilon M_W^2 (M_W^2 - 1)^{-1/2}\delta(\sigma - \sigma_G) = s_+\delta(\sigma - \sigma_G)$ in the supersonic case on the remaining part of the contour. The coefficient $b(\tau)$ and parameters m_0 appearing in it, have the following expressions:

$$b(\tau) = \frac{m_0^2 M (M^2 - 1)^{1/2}\,\mathrm{tg}\,\gamma\,(m_0 - M\cos\tau)\sin\tau}{m m_0^2 A (m_0 - M\cos\tau)^2 - M^2 B\,\mathrm{tg}^2\gamma\,(M - m_0\cos\tau)^2}$$

$$m_0 = [1 - (M^2 - 1)\mathrm{tg}^2\gamma]^{1/2}$$

Relation (2.5) represents the boundary condition of the inhomogeneous Hilbert problem /7, 8/ for the function $\Gamma = \partial p/\partial\sigma - i\partial p/\partial\tau$ with coefficients that are discontinuous at the points A and C.

The solution sought must satisfy two more normalizing conditions (Δv and Δp are the differences in the values of v and p in the regions 2 and 3)

$$B\int_0^\pi \frac{\partial p}{\partial\tau}\frac{d\tau}{y(\tau)} = \Delta v,\quad \int_0^\pi \frac{\partial p}{\partial\tau}d\tau = \Delta p \tag{2.6}$$

the first of which is obtained from the second formula of (2.3), and the second of which is obvious.

The above formulation of the problem is the same as in the case when no heat is supplied to the shock wave; however the heat occurs in the parameters appearing here through the laws of conservation on the detonation front.

3. The pressure on the wall and along the front and the form of the front. The general solution of the boundary value problem formulated here was given in detail in /2/, and is therefore omitted. However, the expressions simplified below for the pressure distribution along the diffracted front and along the wall, reflect the structure of this solution quite well. They have, respectively, the form

$$\frac{\partial p(l + i\tau)}{\partial\tau} = |\Lambda|\frac{b(\tau)L}{\sqrt{b^2(\tau) + 1}}\frac{1 - k\xi}{1 + k\xi}\left[c(\xi - \xi_0) - \frac{c_0}{\sqrt{k}}\frac{\xi^2 + 1}{\xi - \xi_{G,H}}\right] \tag{3.1}$$

$$\frac{\partial p(i\tau)}{\partial\tau} = \Lambda\,\mathrm{Im}(L)\frac{1 - k\xi}{1 + k\xi}\left[c(\xi - \xi_0) - \frac{c_0}{\sqrt{k}}\frac{\xi^2 + 1}{\xi - \xi_{G,H}}\right]$$

The functions L and ξ are given in terms of the elliptic theta functions $\vartheta_1, \ldots \vartheta_4$ whose modulus k depends on the quantity q (2.4)

$$k^2 = 1 - k'^2,\quad 2Kk' = \pi(1 - 2q + 2q^4 - 2q^9 + \ldots)^2$$

$$2K = \pi(1 + 2q + 2q^4 + 2q^9 + \ldots)^2$$

In the first expression of (3.1) the functions L and ξ have $l + i\tau$ as the argument at the image of the front, and in the second expression the quantity $i\tau$ at the image of the wall. The quantity σ_G serves as an argument on the image of the Mach arc (point G) in both expressions. The above functions are given by the formulas

$$L(l + i\tau) = k^{1/2}\frac{\vartheta_2}{\vartheta_1},\ \mathrm{Im}\,L(i\tau) = -k^{1/2}\frac{\vartheta_3}{\vartheta_4},\ \mathrm{Im}\,L(\sigma) = -k^{1/2}\frac{\vartheta_3}{\vartheta_2}$$

$$\xi(l + i\tau) = -k^{-1/2}\frac{\vartheta_3}{\vartheta_2},\quad \xi(i\tau) = -k^{-1/2}\frac{\vartheta_2}{\vartheta_3},\quad \xi(\sigma) = -k^{-1/2}\frac{\vartheta_4}{\vartheta_3}$$

We have for the arguments $l + i\tau$ and $i\tau$ of the functions L and ξ, the corresponding dependence of the theta function on τ, q, and for the argument σ of σ, q' $(\ln q \ln q' = \pi^2)$. The constant c_0 and the quantities constituting it are given by the formulas

$$c_0 = \frac{k^{1/2}s_\pm|\xi_{\sigma,\tau}(\sigma_G, \tau_H)|}{\pi\Phi_\pm(\xi_{G,H})(\xi_{G,H}^2 + 1)},\quad s_\pm = \frac{\varepsilon M_W^2}{\sqrt{\pm 1 + M_W^2}},$$

$$\Phi_\pm = -\mathrm{Im}\,\Phi(\xi_{G,H})$$

$$\Phi = \Lambda L\frac{1 - k\xi}{1 + k\xi},\quad \xi_\sigma' = -\frac{2Kk'}{\pi k^{1/2}}\frac{\vartheta_1\vartheta_2}{\vartheta_3^2},\quad \xi_\tau' = -\frac{2Kk'}{\pi k^{1/2}}\frac{\vartheta_1\vartheta_4}{\vartheta_3^2}$$

Here the theta functions in the expression for ξ_τ' depend on τ, q, and in the expression for ξ_σ' on σ, q'. The subscripts G and H on ξ indicate the transforms of the points to which the quantity ξ refers, and the subscripts σ and τ are variables over which the differentiation is carried out. The plus and minus subscripts correspond to the subsonic and supersonic case respectively.

The function Λ is given, according to /2/, at any point of the region (2.4) or of its contour, by the series

$$\Lambda(z)=\exp\sum_{n=1}^{\infty} g_n \frac{\operatorname{ch} nz}{\operatorname{sh} nl}, \quad g_n=-n^{-1}[4-(F_1{}^n+F_2{}^n)-(F_3{}^n+F_4{}^n)]$$

The sums $F_1{}^n + F_2{}^n$ can be obtained from the recurrent formulas

$$F_1{}^n+F_2{}^n=[(F_1^{n-1}+F_2^{n-1})(H^2-1)-2H\Delta_{n-1}(1-D_1{}^2)]G, \quad n\geqslant 2 \tag{3.2}$$

$$\Delta_{n-1}=[\Delta_{n-2}(H^2-1)+2H(F_2^{n-2}+F_2^{n-2})]G_1^{-1}, \quad n\geqslant 3$$

$$F_1+F_2=2(H^2-1)/G_1, \quad \Delta_1=4H/G_1$$

$$G_{1,2}=H^2+1+2HD_{1,2}, \quad H=[(M+m_0)/(M-m_0)]^{1/2}$$

Formulas (3.2) will also yield the sum $F_3{}^n + F_4{}^n$, if we replace F_1 everywhere by F_3, F_2 by F_4, G_1 by G_2 (the quantity Δ_i can be denoted by $\bar{\Delta}_i$), and D_1 by D_2.

The quantities $D_{1,2}$ are given in terms of the initial parameters of the problem by the relations /2/

$$D_{1,2}=\frac{1-m^2\mp[(1-m^2)^2-4mB[(1-m^2)A-mB]]^{1/2}}{2[(1-m^2)A-mB]}$$

The coefficients c and ξ_0 defined by conditions (2.6) and the normalizing parameters c_1 and c_2 on which they depend, have the following expressions:

$$c=I^{-1}[c_1I_4-c_2I_2-c_0(I_6I_4-I_5I_2)], \quad I=I_1I_4-I_2I_3$$

$$\xi_0(z_0)=-k^{-1/2}c^{-1}I^{-1}[c_1I_3-c_2I_1-c_0(I_6I_3-I_5I_1)]$$

$$I_1=\int_0^{\pi}\vartheta_3\Psi\frac{d\tau}{y}, \quad I_2=\int_0^{\pi}\vartheta_2\Psi\frac{d\tau}{y}, \quad I_3=\int_0^{\pi}\vartheta_3\Psi\,d\tau$$

$$I_4=\int_0^{\pi}\vartheta_2\Psi\,d\tau, \quad I_5=\int_0^{\pi}\frac{\vartheta_2\Psi(\xi^2+1)}{\xi-\xi_{G,H}}d\tau, \quad I_6=\int_0^{\pi}\frac{\vartheta_2\Psi(\xi^2+1)}{\xi-\xi_{G,H}}\frac{d\tau}{y}$$

$$\Psi=\frac{\Lambda b}{\vartheta_1(b^2+1)^{1/2}}\frac{1-k\xi}{1+k\xi}, \quad c_1=\frac{v_+-v_-}{B}=\frac{\Delta v}{B}, \quad c_2=p_+-p_-=\Delta p$$

Here the theta functions depends on τ, q, the functions Λ and ξ on $l+i\tau$, the functions Ψ, b, y and τ and the plus and minus subscripts refer to the right- and left-hand wall separated by the break. The relation $y(\tau)$ appearing here has the form

$$y=m\operatorname{tg}\theta, \quad \operatorname{tg}\theta=\frac{m_0}{M\operatorname{tg}\gamma}\frac{m_0-M\cos\tau}{M-m_0\cos\tau}$$

The components of the velocity $v_\pm$ and pressure $p_\pm$ perturbations in regions *2* and *3* can be found from the formulas (2.2) after we have satisfied ourselves with their linear dependence on the break ε, or we can find them from the exact formulas by carrying out independent calculations of the reflection from the wall for each region in question.

Assuming that the basic solution corresponds to either region *2* or *3* or to some intermediate regions corresponding to intermediate values of the angle of incidence α, we can find the perturbations in question from the following exact formulas:

$$p_\pm=\varkappa_2{}^{-1}(p_{2\pm}/p_1)/(p_{00}/p_1) \tag{3.3}$$

$$v_\pm=M_{W\pm}(a_{2\pm}/a_2)\cos(\gamma-\varepsilon_\pm)-M_W\cos\gamma$$

The subscript 00 indicates the basic flow, and $\varepsilon_\pm$ are the semi-angles of the breaks in the right and left half-wall relative to the wall at which we have the main flow; $\varepsilon_+>0$ if the right half-wall is turned clockwise, and $\varepsilon_->0$ if the left half-wall is turned anticlockwise.

The initial coordinate r at the wall is found from the formula $r=|(M\cos\tau-1)/(M-\cos\tau)|$, and is situated on *ED* (if $\tau>\arccos M^{-1}$) or on *EF* (Fig.1).

The differential equation describing the diffracted segment of the detonation front is obtained by solving the second relation of (2.2) for f', differentiating the result with respect to y and replacing $\partial v/\partial y$ according to (2.3)

$$f''_{yy} = -\frac{FB}{y}\frac{\partial p}{\partial y}, \quad F = \frac{\varkappa_1}{\varkappa_2}\frac{a_2}{a_1}\frac{(\varkappa_2+1)M_d}{(M_d^2-1)(1+\iota^2)} \tag{3.4}$$

Integration with the conditions $f(-m') = f_y'(-m') = 0$ where $\pm m' = \pm\sqrt{1-m^2}$ are the ends of the segment, of the transform of the diffracted segment of the detonation front (the basic flow coincides with the stream in region 2) and simple reduction, yield the function f in two forms

$$f = -FB\int_{-m'}^{y}(y-t)\frac{1}{t}\frac{\partial p}{\partial t}dt = -FBy\int_{-m'}^{y}\frac{1}{t^2}p(t)\,dt \tag{3.5}$$

4. **Simplification of the results.** The study of detonation waves resembling in behaviour the Chapman-Jouget waves, encounters difficulties at the stage of deriving the conditions at the image of the detonation front (2.2). When linearizing the problem in ε, we must expand the square root

$$\sqrt{1+N\frac{f-yf'}{\iota^4}} = \sqrt{1+\bar N}, \quad N = \frac{2}{M_D}\frac{a_2}{a_1}\left(\frac{M_d^2+1}{M_d^2-1}+\iota^4\right) \tag{4.1}$$

occurring in the expressions for the x component of the velocity and pressure perturbations, where the functions $f-yf'$ of the order of ε is determined by the solution of the problem, in a Taylor series.

In order to estimate the possibility of carrying out the above expansion, and hence the admissibility of the solution for small ι, for convenience of the analysis, we first simplified it by expanding it in a series in powers of ι. It is important that although the expansion was carried out in the neighbourhood of $\iota = 0$, the solutions can be used at values of ι near, but not equal to it. The first power of ι is retained in the solution. Omitting all intermediate calculations, we will give the result of simplifying the solution.

The pressure on the wall is independent of ι for small ι. Its distribution has the form (we conditionally assume that $r<0$ at the half-wall *DE* (Fig.1))

$$p(r) = p_+ - \frac{\Delta v}{\pi}\frac{(1-r)^{1/2}}{1+r\sin\gamma} + \frac{\Delta p}{\pi}\arccos\frac{\sin\gamma+r}{1+r\sin\gamma} - \frac{\varepsilon}{\pi}\frac{M_W^2}{(1-M_W^2)^{1/2}}\ln\left|\frac{1-rM_W+(1-r^2)^{1/2}(1-M_W^2)^{1/2}}{r-M_W}\right| \tag{4.2}$$

The pressure varies along the curved segment of the detonation front according to the law (the direction of the y axis corresponds to that in Fig.1)

$$p(y) = p_+ - \frac{1}{i}\frac{\Delta v}{\pi m''}\frac{M_d+1}{M_d-1}\times\left[(1-y'^2)^{1/2} - \frac{\operatorname{arctg}[(M_d^4-1)(1-y'^2)]^{1/2}}{(M_d^4-1)^{1/2}}\right] + J(y') \tag{4.3}$$

$$y' = \frac{y}{m'}, \quad m''^2 = (\varkappa_2+1)\frac{M_d^2-1}{\varkappa_1 M_D^2+1}$$

The term $J(y')$ (or the higher, zero order in ι) is given by the formulas

$$J = -\frac{\Delta p}{\pi}(M_d^2-1)^{-2}(J_1-J_2+J_3)$$

$$J_1 = (M_d^4+1)\arccos(-y'), \quad J_2 = (M_d^4-1)\,y'\,(1-y'^2)^{1/2}$$

$$J_3 = 2M_d^2\,[\operatorname{arctg}[M_d^2 y'^{-1}(1-y'^2)^{1/2}] - j\pi], \quad j = \begin{cases}0, & y'<0\\ 1, & y'>0\end{cases}$$

The form of the difracted detonation front obtained by simplifying (3.5), becomes

$$f(y) = \iota\frac{\Delta v}{\pi}m''\frac{M_d^2F}{M_d^2-1}\left\{y'\left[M_d^{-2}\operatorname{arctg}\left(M_d^{-2}\frac{y'}{1-y'^2}\right) - \arcsin y' - \frac{\pi}{2}\frac{M_d^2-1}{M_d^2}\right] + \frac{\operatorname{arctg}[(M_d^4-1)(1-y'^2)]^{1/2}}{(M_d^4-1)^{1/2}} - (1-y'^2)^{1/2}\right\}$$

and we must omit ι^2 from the expression for F from (3.4).

We see from expression (4.1) that the following expression is necessary for estimating the admissibility of the solution:

$$f - yf' = \iota \frac{\Delta v}{\pi} m'' \frac{M_d^2 F}{M_d^2 - 1)} \left[\frac{\operatorname{arctg} [(M_d^4 - 1)(1 - y'^2)]^{1/2}}{(M_d^4 - 1)^{1/2}} - (1 - y'^2)^{1/2} \right] \tag{4.4}$$

5. Domain of admissibility of the solution.

It is desirable to represent this domain as clearly and explicitly as possible. The non-linear normalization of the solution represents, in this case, a substantial obstruction. The value of the criterion $\bar{N}$ of any solution can only be found if the particular version is known. We must have the function $f - yf'$ with Δv as a factor. It is therefore natural that we should turn to the linear relation connecting Δv and ε, using the formula (4.4) to obtain $f - yf'$.

We can find Δv from the second formula of (2.2) if we know the inclinations of the front at the extremal points of its curved segment, and these can be obtained using Cardan's formulas to solve a cubic equation. The linearization here is found to be very cumbersome and unsuitable for use in the most important symmetric case when the plane of symmetry of the blunt wedge formed by the break in the wall is perpendicular to the front of the incident shock wave. For this reason the boundary of the domain of admissibility is determined, in the symmetric case, by establishing a relation connecting Δv with ε directly and independently of (2.2).

In the present case, having obtained the expression for the velocity of the reflected wave front relative to the wall in the form $U_D - V_1 \cos(\varepsilon + \varepsilon')$ where ε' is the angle between the front and the wall, we can write the conditions of regular reflection and of conservation of the velocity component tangent to the reflected front, in the form

$$[U_D - V_1 \cos(\varepsilon + \varepsilon')] \operatorname{cosec} \varepsilon' = U_0 \operatorname{cosec} \varepsilon, \quad V_1 \sin(\varepsilon + \varepsilon') = v' \cos \varepsilon'$$

and this will yield the formulas

$$\Delta v \frac{a_2}{a_1} = \frac{v'}{a_1} = M_1 \left(1 + \frac{M_D - M_1}{U_0/a_1}\right) \varepsilon \tag{5.1}$$
$$M_1 = (M_0^2 - 1) \left[\left(\varkappa_1 M_0^2 - \frac{\varkappa_1 - 1}{2}\right) \left(1 + \frac{\varkappa_1 - 1}{2} M_0\right) \right]^{-1/2}$$

The expression for the quantity M_D which is necessary here is obtained from the third relation of (1.3), and the magnitude of the formally determined velocity V_2 is found to be the sum of V_1 and of the quantities of second order of smallness. After solving the resulting relation for M_D, we can write the latter in the form

$$M_D = \frac{\varkappa_2 + 1}{2} M_1 + \left(\frac{\varkappa_2 + 1}{2} M_1^2 + \frac{\varkappa_2}{\varkappa_1}\right)^{1/2} \tag{5.2}$$

Finally we find from (1.3) $U_0/a_1 = M_0 a_0/a_1$.

To find the condition of admissibility, we must choose y to be equal to zero, since $|f - yf'|$ has a maximum at $y = 0$: $(f - yf')' = -yf'' = 0$ and the derivative f'' has no zeros at $-m' < y < m'$. From (4.4) we see that $f - yf' = 0$ when $y = \pm m'$, therefore the maximum $|f - yf'|$ will be the largest value in the interval in question. The boundary of the domain of admissibility is found from (4.1) and the condition

$$N(f - yf')/\iota^4 = \bar{N} = \lambda \tag{5.3}$$

We can take for the concave and convex breaks $\lambda = 1$ and $\lambda = -0.5$ according to the sign of $f - yf'$ (see (4.1)). The relative error in computing the roots $(1 \pm \lambda)^{1/2} \approx 1 \pm \lambda/2$ is found to be the same and equal to $3\sqrt{2}/4 \approx 0.06$.

Substituting successively into (5.3) $f - yf' |_{y=0}$ from (4.4), $\Delta v a_2/a_1$ and M_1 from (5.1) and M_D (5.2), neglecting the higher orders of ι in the expressions for N (4.1) and F (3.4) and solving the resulting relation for ι^3/ε, we obtain the boundary of the domain of admissibility (equality sign) and the domain itself

$$\frac{\iota^3}{\varepsilon} \leqslant \frac{2}{\pi} \frac{\varkappa_1}{\varkappa_2} m' \frac{(\varkappa_1 M_D^2 + 1)(M_d^2 + 1)}{\lambda (M_d^2 - 1)^3} \left[\frac{\operatorname{arctg}(M_d^4 - 1)}{M_d^4 - 1} - 1 \right] \times \tag{5.4}$$
$$\left[M_1 + \frac{2}{\varkappa_1 + 1} \frac{M_0^2 - 1}{M_0^3} (M_D - M_1) \right]$$

Fig.3 shows the graph of the inverse of (5.4) against M_0, and three boundaries of admissibility in coordinates ι, ε° for values of M_0 equal to 1.3, 1.5 and 5, with the latter value practically valid for all $M_0 > 5$. For small values of ι they are extended 5 times along the ε° axis. The solutions are admissible to the right of, and below these boundaries. In the case of convex breaks $(\lambda = 0.5)$ the admissible angle of break is twice as small as that in the case of a concave break ($\lambda = 1$).

6. Some conclusions. The second, basic term in the expression for pressure distribution along the detonation front (4.3) is of the order of -1 in ι, that this implies that the solution, in the case when ε and ι are mutually independent, can be real only for finite ι, i.e. when the waves are supercompressed.

However, as soon as we approach the Chapman-Jouget conditions at small ε, the quantity ε/ι^3, according to the condition of admissibility (5.4), will be of the order of unity and the factor $\Delta v/\iota$ will ensure that this term will be of second order in ι. The solution will remain valid.

When considering the symmetric case, we see at once that the total pressure perturbation along the diffracted segment of the front is independent of ι. Integrating the right-hand side of (4.3), we obtain

$$\int_{-m'}^{m'} [p(y) - p_+]\, dy = -\frac{1}{2}\Delta v$$

The distribution of the pressure perturbation along the wall (4.2) is independent of ι for small ι. Both of these facts are true, since we can show that M_W, Δv, Δp and $\gamma \sim O(\iota^2)$.

For the symmetric case ($\alpha = \gamma = 0$) we can obtain this from formula (3.3) by writing the expression for a_1/a_2 from (1.4), as well as from (1.3). When $\gamma \neq 0$, we must bring in a cubic equation for $\operatorname{tg}(\alpha + \gamma)$ /3/. Treating this as an implicit function γ of ι, we can see that $\gamma_\iota' = 0$. However, in /3/ we have a heat supply instead of ι, i.e. we must carry out the substitution according to (1.1) and (1.2).

In the special, symmetric case, the distribution in question is identical with one given in /1/ for acoustic waves. When there is no symmetry, other terms appear corresponding to a flow of finite velocity and finite pressure difference at the walls $\Delta p \neq 0$, past an obtuse wedge.

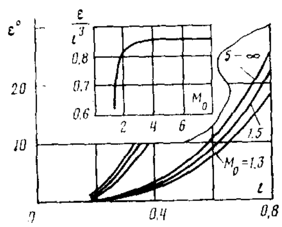

Fig.3

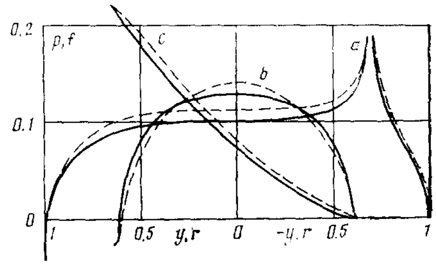

Fig.4

The condition of admissibility and analysis of the calculations for a number of diffraction versions show, that a range of values of ι exists, in which the parameter is not so small that the solution will become invalid for a given ε, but sufficiently small for the straightforward solution linearized over ι to agree satisfactorily with the initial solution.

The last statement is illustrated in Fig.4, which gives a comparison of the simplified solution (the dashed lines) and the non-simplified solution (the solid lines) for the case when the detonation wave is obviously supercompressed: $\iota = 0.619$. The (concave) angle of break is equal to $\varepsilon = -9°$ and we have $Q = 2.1$; $p/p_0 = 8.33$; $p_2/p_1 = 5.76$; $\alpha = 26.5°$. The quantity $\bar{N}$ is equal to 0.978, i.e. the set of initial parameters corresponds to a position near the boundary of admissibility. Curves a and b depict the pressure at the wall and along the front, and c is the shape of the front. When $\iota < 0.25$ and $\varepsilon < 0.5°$, the curves depicting the pressures and the shape of the front, practically coincide.

The authors thank A.A. Barmin for useful comments and discussions.

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Translated by L.K.

PMM U.S.S.R.,Vol.52,No.5,pp.611-615,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

THE PLANE PROBLEM OF HYDROELASTIC STABILITY FOR A HINGE-SUPPORTED PLATE*

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The plane problem of hydroelasticity on the stability of a hinge-supported plate of infinite span placed in a rigid screen is considered in the case of unilateral flow of an ideal incompressible fluid. An analytical representation is obtained for the matrix elements of the averaged aerodynamic loads. The possibility of using the method reduction in the problem under consideration, i.e., of replacing the infinite determinant by a truncated determinant is investigated. Relations are obtained for the flutter velocity as a function of the hydroelasticity and axial force parameters. The problem under consideration was solved in /1-3/ by different methods, where, by assuming the convergence of the infinite determinant to which application of the Bubnov-Galerkin method leads, consideration was confined to two coordinate functions and the forces acting on the fluid side were determined numerically. Only the boundary of the static stability domain was found.

1. Formulation of the hydroelasticity problem.

We will write the equation of the cylindrical vibrations of a plate extended in the stream direction by forces H as follows:

$$Dw_{xxxx} - Hw_{xx} + \varepsilon h\rho_0 w_t + h\rho_0 w_{tt} = p \tag{1.1}$$

Here $w(x, t)$ and $p(x, t)$ are the plate deflection and the fluid pressure thereon, D is the bending stiffness, ε is the damping coefficient, h is the thickness, and ρ_0 is the specific density of the plate material.

The hinge clamping boundary conditions at the points $x = \pm a$ have the form

$$w = w_{xx} = 0 \tag{1.2}$$

The potential of the perturbed fluid velocities $\Phi(x, z, t)$ satisfies the Laplace equation, the damping condition, and the non-penetration condition

$$\Phi_{xx} + \Phi_{zz} = 0, \quad z \leqslant 0 \tag{1.3}$$

$$\lim_{r_*\to\infty} \nabla\Phi = 0, \quad r_* = \sqrt{x^2 + z^2} \tag{1.4}$$

$$\Phi_z = w_t + Vw_x, \quad x \in [-a; a], \quad z = 0 \tag{1.5}$$
$$\Phi_z = 0, \quad x \notin [-a; a], \quad z = 0$$

Here V is the velocity of unperturbed fluid motion.

Using the representation of a harmonic function in the form of the potential difference of a simple and double layer (for instance /4/), and taking into account that the cosine of the angle between the tangent plane and the normal to the surface $z = w(x, t)$ is small compared with unity, we obtain by virtue of (1.5)

$$\Phi(x, 0, t) = 1/\pi \int_{-a}^{a} (w_t + Vw_x)_{x=x'} \ln \frac{a}{|x - x'|} dx' \tag{1.6}$$

Prikl.Matem.Mekhan.,52,5,783-788,1988